Recall

$$\ell^\infty = \{(x_k) : \sup_k |x_k| < \infty\}$$
$$c_0 = \{(x_k) : \lim_k x_k = 0\}$$

Proposition

c_0 is not complemented in ℓ^∞.

Proof by contradiction: Suppose c_0 is complemented in ℓ^∞.

Claim 1: There exists $(f_n) \in (\ell^\infty)^*$ such that $c_0 = \bigcap_n \ker f_n$

Pf: By assumption, there exists closed $F \subset \ell^\infty$ such that $\ell^\infty = c_0 \oplus F$.

Put $P : \ell^\infty \to F$ be the projection.

Then $c_0 = \ker P$ and P is bounded by closed graph thm.

Γ $(x_n), x \in \ell^\infty$, write $x_n = y_n + z_n$ with $y_n \in F$, $z_n \in c_0$.
$x = y + z$ with $y \in F$, $z \in c_0$

Assume $x_n \to x$ and $y_n = Px_n \to y'$.

Since F is closed, $y' \in F$.

On the other hand, $z_n = x_n - y_n \to x - y'$.

Since c_0 is closed, $x - y' \in c_0$

Now $x = y + z = y' + (x - y')$ with $y, y' \in F$, $z, x - y' \in c_0$.

By uniqueness of decomposition, $y' = y = Px$.

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Let $e_n^*: \ell^\infty \to \mathbb{K}$ be the n-th coordinate functional

and $f_n = e_n^* \circ P$. Then $f_n \in (\ell^\infty)^*$ and $c_0 = \ker P = \bigcap_n \ker f_n$

Claim 2: For any $\alpha \in [0,1] \cap \mathbb{Q}^c$, there exists $N_\alpha \subset \mathbb{N}$
such that $N_\alpha \cap N_\beta$ is finite for any $\alpha \neq \beta$

Pf: Write $[0,1] \cap \mathbb{Q} = (r_n)_{n=1}^\infty$

For any $\alpha \in [0,1] \cap \mathbb{Q}^c$, there exists a subsequence $(r_{n_k})_{k=1}^\infty$ such that $r_{n_k} \to \alpha$

Put $N_\alpha := \{n_k : k = 1, 2, \dots\}$.

For any $\alpha \neq \beta$. Take $\varepsilon < |\alpha - \beta|/4$.

$N_\alpha := \{n_k : k = 1, 2, \dots\}$, $N_\beta = \{n'_\ell : \ell = 1, 2, \dots\}$

For k, ℓ large enough, $|n_k - \alpha| < \varepsilon$, $|n'_\ell - \beta| < \varepsilon$.

$|n_k - n'_\ell| \geq |\alpha - \beta| - |n_k - \alpha| - |n'_\ell - \beta| > |\alpha - \beta|/2 > 0$.

Therefore $N_\alpha \cap N_\beta$ is finite.

Claim 3: Put $\chi_\alpha(k) = \begin{cases} 1, & k \in N_\alpha, \\ 0, & k \notin N_\alpha. \end{cases}$

If $f \in (\ell^\infty)^*$ with $c_0 \in \ker f$, then for any $\eta > 0$,
$E = \{\alpha \in [0,1] \cap \mathbb{Q}^c : |f(\chi_\alpha)| \geq \eta\}$ is finite.

Pf: Consider $f = \mathrm{Re}(f) + i\,\mathrm{Im} f$.

We may assume f is real.

Let $\alpha_1, \dots, \alpha_N \in E$.

Put $y_j(k) = \begin{cases} 1, & k \in N_{\alpha_j} \setminus \bigcup_{m \neq j} N_{\alpha_m}, \\ 0, & \text{otherwise}. \end{cases}$

Then $x_{\alpha_j} - y_j \in c_0$ by Claim 2.

Thus $f(x_{\alpha_j}) = f(y_j)$ for $j = 1, \dots, N$.

And $\{k : y_i(k) = 1\} \cap \{k : y_j(k) = 1\} = \phi$ for $i \neq j$.

Then $\|\sum_{j=1}^{N} y_j\|_\infty = 1$.

Thus $\|f\| \geq f(\sum_{j=1}^{N} y_j) = \sum_{j=1}^{N} f(x_{\alpha_j}) \geq N\eta$.

Therefore, $N \leq \frac{\|f\|}{\eta}$.

Proof of Proposition:

Let $S := \bigcup_{n=1}^{\infty} \{\alpha \in [0,1] \cap \mathbb{Q}^c : f_n(x_\alpha) \neq 0\}$

By Claim 3, S is countable.

But $[0,1] \cap \mathbb{Q}^c$ is uncountable.

Then there exists $\gamma \in [0,1] \cap \mathbb{Q}^c$ such that

$x_\gamma \in \bigcap_n \ker(f_n) = c_0$

But $x_\gamma \notin c_0$ since N_γ is infinite.

Contradiction!

$\square$